UDC 534

STEADY-STATE SOLUTIONS OF A DISPERSIVE SYSTEM OF NONLINEAR WAVES

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An "averaged Lagrangian" method of obtaining a dispersive system describing slow variations of the wave parameters for quasi-stationary waves, is obtained for equations admitting the existence of wave solutions. In the first "adiabatic" approximation the dispersive system for the Klein-Gordon equation represents a quasi-linear system of the hyperbolic type and admits discontinuous solutions. The structure of the discontinuities for the conservative and the nonconservative cases is investigated and the number of free parameters in a discontinuity determined.

Various asymptotic methods find wide application in investigating nonlinear waves with dispersion [1, 2]. An adiabatic approximation method of obtaining a dispersive system of equations was proposed in [3] for the quasi-stationary waves, i.e. for the waves in which the change in the wave parameters is slow compared with the fundamental oscillations. That method is based on the process of averaging over the fast variable appearing in equations equivalent to the initial equations and written in divergent form. It was shown first for the conservative systems [4] and then for the nonconservative ones [5] that the dispersive system can be obtained from a variational equation averaged in the appropriate manner. Such averaging processes representing integration over a part of the independent variables find use in the theory of stress and strain in shells and rods when the Bubnov method is used, and also in the asymptotic theory of nonlinear oscillations [6].

1. Derivation of a dispersive system from a variational equation in the averaged form. We consider the following Euler-type equation

$$-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right)-\frac{\partial}{\partial x}\left(\frac{\partial L}{\partial u_x}\right)+\frac{\partial L}{\partial u}+Q_{(u)}=0 \qquad (1.1)$$

for the generalized variational equation [7]

$$\delta\int_{V_0} L(u, u_t, u_x)\,dx\,dt+\delta W^*=0 \qquad (1.2)$$

where δW^* is a nonholonomic variation of the functional

$$\delta W^*=\int_{V_0} Q_{(u)}(u, u_t, u_x, \varepsilon)\,\delta u\,dx\,dt, \qquad Q_{(u)}=\varepsilon Q_{(u)}^{(1)}+\varepsilon^2 Q_{(u)}^{(2)}+\ldots$$

and ε is a small, positive parameter. In (1.2), the variations at the boundary V_0 of the region of integration are equated to zero.

Let us assume that for $\varepsilon=0$ Eq. (1.1) has a solution in the form of a travelling stationary wave with constant parameters a, $\omega^{(0)}$, $k^{(0)}$

$$u = U^{(0)}(\vartheta, a), \qquad \vartheta = k^{(0)}x - \omega^{(0)}t$$

The quasi-stationary wave solutions of (1.1) are characterized by a slow change with respect to x and t of the parameters a, ω and k. Let us assume that these solutions can be represented as expansions in ε of the form

$$u = \sum_{m=0} \varepsilon^m U^{(m)}(\vartheta, a, \ldots) \tag{1.3}$$

$$\vartheta_t \equiv -\omega = -\sum_{m=0} \varepsilon^m \omega^{(m)}(\varepsilon x, \varepsilon t), \qquad \vartheta_x \equiv k = \sum_{m=0} \varepsilon^m k^m(\varepsilon x, \varepsilon t)$$

$$a_t = \sum_{m=0} \varepsilon^{m+1} A^{(m+1)}(\varepsilon x, \varepsilon t), \qquad a_x = \sum_{m=0} \varepsilon^{m+1} B^{(m+1)}(\varepsilon x, \varepsilon t)$$

Applying the usual method of obtaining $U^{(m)}$, we substitute (1.3) into (1.1) and equate to zero the coefficients of like powers in ε to obtain, from the conditions of periodicity of the functions $U^{(0)}$, $U^{(1)}$, . . . , a dispersive partial differential system in a, ω and k [5, 8]. The problems of convergence in ε are not considered here, the solutions studied being of prescribed accuracy with respect to ε. The same dispersive system can be obtained from the averaged variational equation using the variations $\delta\vartheta$ and δa. The averaged Lagrangian $\bar{L}$ is introduced in the form

$$\bar{L}(\vartheta_t, \vartheta_x, a, a_t, a_x, H_t, H_x, H) = \frac{1}{2\pi}\int_0^{2\pi} L(U^{(0)}, \vartheta_t U_\vartheta^{(0)} + a_t U_a^{(0)} + H_t U_H^{(0)}, \ldots)\, d\vartheta \tag{1.4}$$

where $U_\vartheta^{(0)}$ and $U_a^{(0)}$ are derivatives with respect to ϑ and a appearing in the explicit form, and H incorporates the remaining terms which change slowly and are not subject to further variation. The averaged nonholonomic variation $\overline{\delta W^*}$ is given in the form

$$\overline{\delta W^*} = \int_{V_0} [Q_{(\vartheta)}\delta\vartheta + Q_{(a)}\delta a]\, dx\, dt \tag{1.5}$$

$$Q_{(\vartheta)} = \frac{1}{2\pi}\int_0^{2\pi} Q_{(u)}(u, u_t, u_x, \varepsilon)\, U_\vartheta^{(0)}\, d\vartheta, \qquad Q_{(a)} = \frac{1}{2\pi}\int_0^{2\pi} Q_{(u)}(u, u_t, u_x, \varepsilon)\, U_a^{(0)}\, d\vartheta$$

Here $Q_{(\vartheta)}$ and $Q_{(a)}$ are the averaged generalized forces and the function u is taken in the form of (1.3). We note that for the conservative systems when $\delta W^* = 0$, the stationary solution $U^{(0)}$ is sufficient to define the dispersive system in any degree of approximation with respect to ε.

The next approximation for the dispersive system, following the stationary one, may be obtained by equating to zero a_t and a_x in (1.4), i.e. taking the averaged Lagrangian in the form

$$\bar{L} = \bar{L}^{(0)}(\vartheta_t, \vartheta_x, a)$$

and setting $u = U^{(0)}$ in (1.5). The resulting quasi-linear system describes the adiabatic approximation due to Whitham [3, 9].

In a number of important cases this quasi-linear system is hyperbolic and admits the existence of discontinuous solutions. These discontinuities, which were brought into consideration by Whitham [3] are, unlike the shock waves in the continuous media, not connected with the irreversible transformations of energy and provide schemata for the domains of rapid variation of the wave parameters in the first approximation. The dispersive system in this approximation is not valid for these narrow zones, but can be used to record the conditions at the discontinuities. These conditions can be obtained from both the dispersive system written in the divergent form and the averaged variational equation [10], and they have the form

$$\left[\overline{L}^{(0)} - k\frac{\partial \overline{L}^{(0)}}{\partial k}\right] = v\left[k\frac{\partial \overline{L}^{(0)}}{\partial \omega}\right]$$
$$\left[\omega\frac{\partial \overline{L}^{(0)}}{\partial k}\right] = v\left[\overline{L}^{(0)} - \omega\frac{\partial \overline{L}^{(0)}}{\partial \omega}\right] \tag{1.6}$$

where v is the rate of propagation of the discontinuity and $[\varphi]$ denotes the difference in the values of the function φ ahead and behind of the discontinuity. We note that the conditions (1.6) refer to the first adiabatic approximation.

In a higher order approximation the dispersive system makes possible the study of the structure and evolutionarity of these discontinuities.

2. Conservative systems. As an example, we consider the Klein-Gordon equation

$$u_{tt} - u_{xx} + V'(u) = 0 \tag{2.1}$$

for which in the variational equation (1.2) we have

$$L = -\frac{u_t^2}{2} + \frac{u_x^2}{2} + V(u), \qquad \delta W^* = 0$$

and $V'(u)$ is a nonlinear function. The stationary wave $u^{(0)}(\vartheta,\ a)$ is determined by the solution of the following ordinary differential equation:

$$(\omega^{(0)^2} - k^{(0)^2})\,U^{(0)}_{\vartheta\vartheta} + V'(U^{(0)}) = 0$$

the first integral of which has the form

$$\frac{\omega^{(0)^2} - k^{(0)^2}}{2}\,U^{(0)^2}_{\vartheta} + V(U^{(0)}) = \frac{a^2}{2}$$

and can be obtained by inverting the integral

$$\vartheta = \pm\sqrt{\omega^{(0)^2} - k^{(0)^2}}\int\limits_{a_1}^{U^{(0)}}\frac{dy}{\sqrt{a^2 - 2V(y)}} \tag{2.2}$$

Here a_1 and a_2 denote two consecutive roots of the equation

$$s(y) \equiv a^2 - 2V(y) = 0$$

such that when $a_1 < y < a_2$ we have $s(y) > 0$ and the modulus of $s(y)$ is of the order of $|y - a_i|^{\alpha_i}$ near a_i when $\alpha_i < 2$. It can be seen from (2.2) that $U^{(0)}$ is a function even in ϑ. In the following we shall consider the branch corresponding to $\vartheta > 0$.

Let us write the averaged Lagrangian (1.4) for Eq. (2.1)

$$\overline{L} = \frac{1}{2\pi}\int\limits_0^{2\pi}\left[\frac{-\vartheta_t^2 + \vartheta_x^2}{2}U^{(0)^2}_{\vartheta} + \frac{-a_t^2 + a_x^2}{2}U^{(0)^2}_{a} + V(U^{(0)})\right]d\vartheta \equiv$$
$$\frac{-a_t^2 + a_x^2}{2}\varphi_1 + \frac{-\vartheta_t^2 + \vartheta_x^2}{2}\varphi_2 + \varphi_3$$

(the functions φ_1, φ_2 and φ_3 are determined as respective integrals of $U_a^{(0)2}$, $U_\vartheta^{(0)2}$ and $V(U^{(0)})$ Then the dispersive system is found to be composed of the following equations

$$\frac{\partial}{\partial t}(\omega\varphi_2) + \frac{\partial}{\partial x}(k\varphi_2) = 0 \tag{2.3}$$

$$a_{tt} - a_{xx} - \frac{a_t^2 - a_x^2}{2}\frac{\varphi_{1a}}{\varphi_1} - \frac{\omega^2 - k^2}{2}\frac{\varphi_{2a}}{\varphi_1} + \frac{\varphi_{3a}}{\varphi_1} = 0 \tag{2.4}$$

$$\omega_x + k_t = 0 \tag{2.5}$$

Equations (2.3) and (2.4) can be written in the form of the conservation laws following from the Noether theorem [11] on invariance of the Lagrangian $\bar{L}$ under the group of translations in t and x

$$\frac{\partial T_{11}}{\partial t} + \frac{\partial T_{12}}{\partial x} = 0, \qquad \frac{\partial T_{21}}{\partial t} + \frac{\partial T_{22}}{\partial x} = 0 \tag{2.6}$$

where

$$T_{11} = a_t\bar{L}_{a_t} + \vartheta_t\bar{L}_{\vartheta_t} - \bar{L} = -\frac{a_t^2 + a_x^2}{2}\varphi_1 - \frac{\vartheta_t^2 + \vartheta_x^2}{2}\varphi_2 - \varphi_3$$

$$T_{12} = a_t\bar{L}_{a_x} + \vartheta_t\bar{L}_{\vartheta_x} = a_t a_x\varphi_1 + \vartheta_t\vartheta_x\varphi_2$$

$$T_{21} = a_x\bar{L}_{a_t} + \vartheta_x\bar{L}_{\vartheta_t} = -T_{12}$$

$$T_{22} = a_x\bar{L}_{a_x} + \vartheta_x\bar{L}_{\vartheta_x} - \bar{L} = \frac{a_t^2 + a_x^2}{2}\varphi_1 + \frac{\vartheta_t^2 + \vartheta_x^2}{2}\varphi_2 - \varphi_3$$

In the adiabatic approximation the second order terms in ε in the dispersive system should be neglected. Then (2.5) yields the following dispersive relation connecting ω, k and a

$$\omega^2 - k^2 = \frac{2\varphi_{3a}}{\varphi_{2a}} \equiv \Phi(a) \tag{2.7}$$

from which we can find $\omega(k, a)$ whereupon the complete dispersive system becomes

$$\frac{\partial}{\partial t}\left[\sqrt{k^2 + \Phi(a)}\,\varphi_2(a)\right] + \frac{\partial}{\partial x}\left[k\varphi_2(a)\right] = 0 \tag{2.8}$$

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}\left[\sqrt{k^2 + \Phi(a)}\right] = 0$$

This dispersive system is a hyperbolic, quasi-linear, reducible system and admits two families of characteristics [3]. It leads to multivalued solutions, and its discontinuous solutions with conditions (1.6) at the discontinuity were considered in [3]. The necessary condition for the discontinuity in the system (2.8) to be evolutionary [12] is the presence of three characteristics at it. This condition forms a system of three inequalities, their corollaries were considered by Whitham in [3]. We note that in the adiabatic approximation the conditions at the discontinuity leave three parameters unrestricted.

In the linear case $V = u^2/2$ Eq. (2.1) becomes the telegraph equation, and the dispersive relation (2.7) is independent of the amplitude and $\Phi(a) = 1$. In this case the characteristic directions coincide and are equal to the group velocity $d\omega/dk$, therefore the number of characteristics at the discontinuity is not greater than two and the discontinuity is not evolutionary. This result is obvious, by virtue of the principle of superposition of solutions.

To clarify the structure and the conditions of evolutionarity of the discontinuity, we consider the complete dispersive system (2.3) – (2.5). We seek its solutions in the form of a function of the variable $\xi = x - vt$. From (2.3), (2.5) and (2.6) we obtain a system of ordinary differential equations the first integrals of which have the form

$$-2v\varphi_3 + v(1 - v^2)\left(\frac{da}{d\xi}\right)^2\varphi_1 - \omega k(1 - v^2)\varphi_2 = v(1 - v^2)c_3$$

$$\omega - vk = c_2, \qquad - v\omega\varphi_2 + k\varphi_2 = c_1 \tag{2.9}$$

Eliminating k and ω and introducing the total energy $E = a^2 / 2$, we obtain an equation for $E = E(\xi)$, which can be solved in quadratures

$$\left(\frac{dE}{d\xi}\right)^2 = \frac{2E}{\varphi_1\varphi_2(1-v^2)^2}\Big\{-\varphi_2^2c_2^2 + 2(1-v^2)\varphi_2\varphi_3 + \varphi_2\left[c_3(1-v^2)^2 - c_1c_2\left(v+\frac{1}{v}\right)\right] - c_1^2\Big\} \equiv F(E) \tag{2.10}$$

For the linear equations $(V' = u)$ the functions φ_i are easily computed $(U^{(0)} = a\cos\vartheta)$

$$\varphi_1 = 1/2, \quad \varphi_2 = E, \quad \varphi_3 = 1/2E$$

Then $F(E)$ in (2.10) is a quadratic trinomial

$$F(E) = \frac{4}{(1-v^2)^2}\left\{E^2[-c_2^2 + (1-v^2)] + E\left[c_3(1-v^2)^2 - c_1c_2\left(v+\frac{1}{v}\right)\right] - c_1^2\right\}$$

Let us consider $F(E)$ on the (F, E)-plane (curve 1 in Fig. 1). A bounded solution is possible if the following two conditions hold:

$$c_2^2 > (1-v^2)$$

$$\left[c_3(1-v^2) - \frac{c_1c_2}{v}(1-v^2)\right]^2 > -4c_1^2(1-v^2) - 4c_1c_2c_3(v^2-2)$$

The solution describes sinusoidal modulation of the amplitude, as well as ω and k. This result agrees with the conclusion that no discontinuities are possible in the linear equation written for the dispersive system in the adiabatic approximation.

Let us now consider the nonlinear case and show the possible forms of the function $F(E)$, defined by Eq. (2.10), which lead to bounded solutions of (2.10).

1). Equation

$$F(E) = 0 \tag{2.11}$$

admits two simple roots $E_1 < E_2$ such that $F(E) > 0$ for $E_1 < E < E_2$. This case is analogous to one already considered for the linear equation and represents the modulations of a, ω and k.

2). In Eq. (2.11) E_1 is a simple root and E_2 is a double root (curve 2 in Fig. 1). Then the solution $E = E(\xi)$ has the form of a single, isolated wave [1], the wave amplitude is equal to $E_2 - E_1$, and the value at infinity is E_2 (isolated rarefaction wave). In this case the conditions (2.9) are supplemented with an additional condition

$$F'(E_2) = 0$$

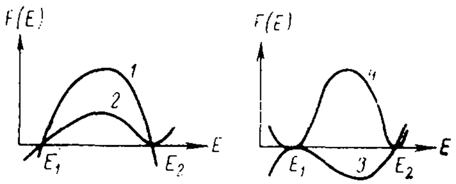

Fig. 1

and three out of seven parameters (E_i, ω_i, k_i, c, ($i = 1, 2$) remain unconstrained.

A single wave-type solution is also obtained in the case when E_1 is a simple root and E_2 is an n-tuple root ($n \leq 5$). Then the additional conditions are $F'(E_2) = \ldots = F^{(n)}(E_2) = 0$ and E tends to E_2 with $\xi \to \pm\infty$ in an algebraic manner. Since we impose n additional restrictions, the number of the unrestricted parameters is equal to $5 - n$.

3). Let in Eq. (2.11) E_1 be a double root and E_2 a simple root (curve 3 in Fig. 1). Then a hydraulic shock type solution is possible, i.e. a discontinuity between E_1 and

E_2. However, the integral curve $E = E_2$ is singular, therefore the hydraulic shock will be stable provided that $E = E_2$ is also a double root. In this case we have two unconstrained parameters (see below).

4). If E_1 and E_2 are both double roots (curve 4 in Fig. 1) and $F(E) > 0$ for $E_1 < E < E_2$, i.e. if the conditions

$$F(E_i) = 0, \qquad F'(E_i) = 0 \qquad (i = 1, 2)$$

hold, then the solution obtained has the form of a smoothly varying amplitude between E_1 and E_2 and the derivatives of E, k and ω tend to zero as $\xi \to \pm\infty$. The first integrals (2.9) form three conditions at the discontinuities and Eq. (2.4) of the dispersive system provides another two conditions

$$\omega_i^2 - k_i^2 = \frac{\varphi_3'(E_i)}{\varphi_2'(E_i)} \qquad (i = 1, 2) \tag{2.12}$$

It can be shown that the requirement that $F'(E_i) = 0$ is equivalent to (2.12). Thus we have five conditions for seven parameters, leaving two parameters unconstrained. We note that the necessary conditions for the existence of a discontinuity in the adiabatic approximation will not be sufficient when discontinuous type solutions are considered in a complete dispersive system.

3. Nonconservative systems. Let us consider the linear Klein-Gordon equation whose right-hand side is not zero, setting

$$L = -\frac{u_t^2}{2} + \frac{u_x^2}{2} + \frac{u^2}{2}, \qquad \delta W^* = \varepsilon \int\limits_{V_0} Q(u, u_t, u_x, \varepsilon)\, \delta u\, dx\, dt$$

In this case the dispersive system has the form

$$\begin{gathered}\frac{\partial}{\partial t}\left(\frac{\omega a^2}{2}\right) + \frac{\partial}{\partial x}\left(\frac{k a^2}{2}\right) - \varepsilon Q_{(\theta)} = 0\\ a_{tt} - a_{xx} + (k^2 - \omega^2 + 1)\, a + 2\varepsilon Q_{(a)} = 0\\ \omega_x + k_t = 0\end{gathered} \tag{3.1}$$

where $Q_{(\theta)}$ and $Q_{(a)}$ are determined in accordance with (1.5).

Following (2.4) we write the equivalent system in the form of the laws of conservation

$$\frac{\partial T_{11}}{\partial t} + \frac{\partial T_{12}}{\partial x} = \varepsilon Q_{(a)} a_t + \varepsilon Q_{(\theta)} \omega, \qquad \frac{\partial T_{21}}{\partial t} + \frac{\partial T_{22}}{\partial x} = \varepsilon Q_{(a)} a_x + \varepsilon Q_{(\theta)} k \tag{3.2}$$

where T_{ij} is given by (2.6) and $\bar{L}$ has the form

$$\bar{L} = \frac{-a_t^2 + a_x^2}{4} + \frac{-\omega^2 + k^2}{4} a^2 + \frac{a^2}{4}$$

At first we shall limit ourselves to the terms of the order not higher than the first in ε. Then (3.1) yields the following dispersive relation equivalent to (2.5):

$$\omega^2 = 1 + k^2 + \varepsilon \frac{2}{a} Q_{(a)}^0 (a, k) \tag{3.3}$$

Let us write the first equation of (3.1) in the same approximation in the form

$$E_t + c(k) E_x + E c'(k) k_x - \frac{\varepsilon}{\omega} Q_{(\theta)}^0 = 0, \quad c(k) = \frac{k}{\sqrt{k^2 + 1}} \tag{3.4}$$

We consider a particular solution of the system (3.2) – (3.4) corresponding to the self-oscillatory mode, for a stationary monochromatic wave

$$\omega = \omega_0 = \text{const}, \quad k = k_0 = \text{const}, \quad E = E_0 = \text{const},$$

assuming that the condition $Q^0_{(\vartheta)} = 0$ must hold, i.e. that a frequency-amplitude dependence exists for the self-oscillatory mode. Let us pose the problem of stability of such a solution. A system written in variations with the accuracy of up to and including terms of order of ε, has the form

$$\frac{\partial (\delta E)}{\partial t} + c(k_0)\frac{\partial (\delta E)}{\partial x} + E_0 c'(k_0)\frac{\partial (\delta k)}{\partial x} - \varepsilon\left[\frac{\partial q}{\partial E}\delta E + \frac{\partial q}{\partial k}\delta k\right] = 0$$

$$\frac{\partial (\delta k)}{\partial t} + c(k_0)\frac{\partial (\delta k)}{\partial x} = 0, \qquad q \equiv \frac{1}{\omega(k_0)} Q^0_{(\vartheta)}(k_0, E_0)$$

Then solution of the second equation is given by

$$\delta k = \psi(x - c(k_0)\, t)$$

where ψ is an arbitrary function. If the solution δE is sought in the form of a function of $x - c(k_0)t$, then it has the form

$$\delta E = D\exp \varepsilon\frac{\partial q}{\partial E}\xi - \int_{\xi_0}^{\xi}\left[c' E_0 \frac{\partial \psi(\eta)}{\partial \eta} - \varepsilon\frac{\partial q}{\partial k}\psi(\eta)\right]\exp\left[\varepsilon\frac{\partial q}{\partial E}(\xi - \eta)\right] d\eta$$

i.e. the stability of the self-oscillatory mode is determined by the sign of $\partial q / \partial E$.

For stable self-oscillatory modes we consider discontinuous solutions, regarding a discontinuity in the sense used above. To clarify the problem of stability of a discontinuity, one must consider the approximation following the adiabatic one, and for this it is sufficient to take the dispersive system in the form

$$-\frac{\partial}{\partial t}\left(\frac{a^2\omega}{2}\right) - \frac{\partial}{\partial x}\left(\frac{a^2 k}{2}\right) + \varepsilon Q_{(\vartheta)} = 0$$

$$\omega^2 - k^2 = 1 - \varepsilon\frac{2}{a} Q_{(a)}(a, \omega, k), \qquad \omega_x + k_t = 0 \tag{3.5}$$

$$Q_{(a)} = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\vartheta, \omega a\sin\vartheta, -ka\sin\vartheta)\cos\vartheta d\vartheta$$

$$Q_{(\vartheta)} = \frac{a}{2\pi}\int_0^{2\pi} f(a\cos\vartheta + \varepsilon U^{(1)}, \omega a\sin\vartheta + a_t\cos\vartheta - \varepsilon\omega U^{(1)}_\vartheta,$$
$$- ka\sin\vartheta + a_x\cos\vartheta + \varepsilon k U^{(1)}_\vartheta)\sin\vartheta d\vartheta$$

where in $Q_{(a)}$ we take into account the terms of the order of ε^0 and in $Q_{(\vartheta)}$ the terms of the order of ε^0 and ε^1, and assume that within the limits of accuracy chosen $Q_{(a)} = Q_{(a)}(a, k)$. When computing $U^{(1)}$, we can express it in the terms of a, ω and k without resorting to functions appearing in the asymptotic representations (1.3). For this reason, in the general case, we have in the second approximation

$$Q_{(\vartheta)} = Q_{(\vartheta)}(\varepsilon, a, a_t, a_x, \omega, k)$$

and $Q_{(\vartheta)}$ can, within the limits of accuracy considered, be written in the form

$$Q_{(\vartheta)} = Q^{(0)}_{(\vartheta)}(a, \omega, k) + a_t Q^{(1)}_{(\vartheta)}(a, \omega, k) + a_x Q^{(2)}_{(\vartheta)}(a, \omega, k)$$

Clearly, the necessary condition for existence of discontinuous solutions is the following

equality: $$Q^{(0)}_{(\vartheta)} = 0 \tag{3.6}$$
which must hold for the admissible values of a, ω and k ahead and behind the discontinuity.

Solving (3.5) and (3.6) jointly, we obtain an expression for $Q_{(a)}$ depending on a only. As a result, the dispersive relation becomes

$$\omega^2 = k^2 + \Phi(a), \qquad \Phi(a) = 1 - 2a^{-1}\varepsilon Q_{(a)}(a) \tag{3.7}$$

The above equation represents a particular case of the dispersive relation for the nonlinear Klein-Gordon equation. If the dispersive system is written out in the form of the laws of conservation (3.2), and (3.6) with (3.7) are taken into account, then its right-hand sides have the form

$$\varepsilon Q_{(a)} a_t - \varepsilon Q_{(\vartheta)}\omega = \varepsilon F'(a)a_t - \varepsilon\omega(a) \times$$
$$[a_t Q^{(1)}_{(\vartheta)}(a,\omega(a), k(a)) + a_x Q^{(2)}_{(\vartheta)}(a, \omega(a), k(a))]$$

$$\varepsilon Q_{(a)} a_x + \varepsilon Q_{(\vartheta)} k = \varepsilon F'(a) a_x + \varepsilon k(a) \times$$
$$[a_t Q^{(1)}_{(\vartheta)}(a,\omega(a), k(a)) + a_x Q^{(2)}_{(\vartheta)}(a, \omega(a), k(a))]$$

where $F(a)$ is the primitive of $Q_{(a)}(a)$. Introducing the primitive functions for the remaining terms $\Phi^{ij}(a)$, we can write the dispersive system in the following divergent form

$$\frac{\partial}{\partial t}\left\{\frac{(\omega^2+k^2)a^2}{4} + \frac{a^2}{4} + \varepsilon F(a) - \varepsilon\Phi^{11}(a)\right\} + \frac{\partial}{\partial x}\left\{\frac{\omega k a^2}{2} - \varepsilon\Phi^{12}(a)\right\} = 0$$
$$\frac{\partial}{\partial t}\left\{\frac{\omega k a^2}{2} - \varepsilon\Phi^{22}(a)\right\} + \frac{\partial}{\partial x}\left\{\frac{(\omega^2+k^2)a^2}{4} - \frac{a^2}{4} - \varepsilon F(a) - \varepsilon\Phi^{21}(a)\right\} = 0$$

From this the obvious conditions at the discontinuity follow, they can be regarded as a generalization of conditions (1.6). We note that the present case differs from that of conservative systems ; here only a single parameter remains unrestricted (owing to the necessity of satisfying the two supplementary conditions (3.6) ahead and behind of the discontinuity).

In connection with the argument given in Sect.2 we find it important to take into consideration the next (third) approximation, as this will unable us to reveal the structure of the discontinuity. We shall seek a solution of the system (3.1) expressed as a function of the variable $\xi = x - vt$, where v is the rate of propagation of the discontinuity and is constant, remembering that $Q_{(a)}$ should include terms of the order of ε, and $Q_{(\vartheta)}$ terms of the order of ε^2 inclusive

$$Q_{(a)} = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\vartheta + \varepsilon U^{(1)}, \omega a\sin\vartheta + a_t\cos\vartheta - \varepsilon\omega U^{(1)}_{\vartheta}, -ka\sin\vartheta +$$
$$a_x\cos\vartheta + \varepsilon k U^{(1)}_{\vartheta})\cos\vartheta d\vartheta$$

$$Q_{(\vartheta)} = \frac{a}{2\pi}\int_0^{2\pi} f(a\cos\vartheta + \varepsilon U^{(1)} + \varepsilon^2 U^{(2)}, \omega a\sin\vartheta + \ldots - \varepsilon^2\omega U^{(2)}_{\vartheta}, -$$
$$ka\sin\vartheta + \ldots + \varepsilon^2 k U^{(2)}_{\vartheta})\sin\vartheta d\vartheta$$

We note that for the linear case the functions $U^{(1)}$ and $U^{(2)}$ can be expressed in terms of a, ω and k, so that

$$Q_{(\vartheta)} = Q_{(\vartheta)}(\varepsilon, a, \omega, k, a_t, a_x, \omega_t, \omega_x, k_t, k_x)$$
$$Q_{(a)} = Q_{(a)}(\varepsilon, a, \omega, k, a_t, a_x)$$

The following conditions must be fulfilled in the course of solution:

1) for $\xi \to +\infty$ the values of a, k and ω ahead of the discontinuity tend to constants a_1, k_1 and ω_1;

2) for $\xi \to -\infty$ the values of a, k and ω behind of the discontinuity tend to constants a_2, k_2 and ω_2;

3) for $\xi \to \pm\infty$, the derivatives of any order in ξ tend to zero.

If such solutions exist, they have the discontinuity character. Then the first two equations of (3.1) yield the following four conditions ($i = 1, 2$)

$$Q_{(\vartheta)}(a_i, k_i, \omega_i, 0, \ldots, 0) = 0 \qquad (3.8)$$
$$a_i(k_i^2 - \omega_i^2 + 1) + 2\varepsilon Q_{(a)}(a_i, k_i, \omega_i, 0, 0) = 0$$

Finally, the last equation of (3.1) gives the first integral

$$\omega - vk = c_2 \qquad (3.9)$$

From (3.1) and (3.2) we obtain the following dispersive system in E and k

$$\frac{d}{d\xi}(kE) - v\frac{d}{d\xi}[E(vk + c_2)] - \varepsilon Q_{(\vartheta)}\left(a, k, vk + c_2, -v\frac{dE}{d\xi}, \frac{dE}{d\xi}, \ldots\right) = 0$$
$$-v\frac{dE}{d\xi} + (1 - v^2)\frac{d}{d\xi}[(vk + c_2)kE] + \frac{d}{d\xi}\left[v(v^2 - 1)\frac{1}{4E}\left(\frac{dE}{d\xi}\right)^2\right] + \qquad (3.10)$$
$$2\varepsilon v\frac{Q_{(a)}}{\sqrt{2E}}\frac{dE}{d\xi} + \varepsilon(c_2 + 2vk)Q_{(\vartheta)} = 0$$

Substituting $dE / d\xi = p$, we reduce this system to its normal form

$$dk/d\xi = k' = f_1(k, E, p), \qquad p' = f_2(k, E, p), \qquad E' = p$$

The latter system is autonomous. Solving the first two equqtions for E, we can obtain a solution for the following initial values:

$$k = k_1 \qquad p = p_1 = 0 \quad \text{when } E = E_1$$

Thus we obtain an equation for $dE / d\xi$

$$\frac{dE}{d\xi} = p(E, E_1, k_1, \varepsilon)$$

which is a function of the initial values. The constants E_1 and k_1 can be overdefined using the first integrals of the system of Eqs. (3.8) and (3.9)

$$(1 - v^2)kE - vc_2E - \varepsilon\int_{E_1}^{E} Q_{(\vartheta)}(\varepsilon, E, vk + c_2, k, -vp, p, \ldots, f_1)\frac{dE}{p} = c_1.$$

$$vE + (v^2 - 1)(c_2 + vk)Ek + \frac{v(v^2 - 1)}{4E}E'^2 + 2\varepsilon v\int_{E_1}^{E} Q_{(E)}\,dE +$$

$$\varepsilon c_2\int_{E_1}^{E}\frac{Q_{(\vartheta)}dE}{p} + 2v\varepsilon\int_{E_1}^{E}\frac{Q_{(\vartheta)}k}{p}\,dE = c_3, \qquad Q_{(E)} \equiv \frac{Q_{(a)}}{\sqrt{2E}}$$

Let us set

$$c_1 = (1 - v^2)k_1E_1 - vc_2E_1, \qquad c_2 = vE_1 + (v^2 - 1)(c_2 + vk_1)E_1k_1$$

This enables us to express k_1 and E_1 by c_1, c_2 and c_3. Moreover, the requirement that $p_1 = dE/d\xi|_{\xi=+\infty} = 0$ is fulfilled in this case automatically.

Let us now set

$$I(E) = \int_{E_1}^{E} Q_{(\theta)} \frac{dE}{p}$$

find k from (3.9) and insert it into (3.10). After some manipulations we obtain

$$\frac{v(v^2-1)}{4}\left(\frac{dE}{d\xi}\right)^2 \equiv \frac{v(v^2-1)}{4} F(E) = c_3 E - vE^2 + \frac{(c_1 + c_2 vE)(vc_1 + c_2 E)}{1 - v^2} -$$

$$2\varepsilon E v \int_{E_1}^{E} Q_{(E)}\, dE + \frac{2vEc_1\varepsilon}{v^2-1}\int_{E_1}^{E} \frac{I}{E^2}\, dE - \frac{vE\varepsilon^2}{1-v^2}\int_{E_1}^{E}\frac{I^2}{E^2}\, dE$$

The necessary conditions for the initial dispersive system to have discontinuity type solutions, are

$$F(E_i) = 0, \qquad F'(E_i) = 0 \qquad (i = 1,2)$$

We have shown before that $F(E_1) = 0$. If the following relations emerging from (3.9) and (3.10) hold

$$(1 - v^2)k_2 E_2 - vc_2 E_2 - \varepsilon I(E_2) = c_1$$

$$vE_2 + (v^2-1)(c_2 + vk_2)E_2 k_2 + 2\varepsilon v\int_{E_1}^{E_2} Q_{(E)} dE + \varepsilon C_2 I(E_2) + 2v\varepsilon \int_{E_1}^{E_2} \frac{Q_{(\theta)} k\, dE}{p} = c_3$$

then the last of them is equivalent to $F(E_2) = 0$. It can be shown that

$$\frac{v(v^2-1)}{4}(F - \varepsilon F') = E^2 v\{1 + k^2 - \omega^2 + 2\varepsilon Q_{(E)}\}$$

Assuming $E = E_i$ we find, on the basis of (3.6) and the equality $F(E_i) = 0$ that in this case $F'(E_i) = 0$.

Thus we have imposed seven following conditions on the seven parameters V, ω_i, k_i and a_i

$$Q_{(\theta)}(k_i, \omega_i, E_i, 0, \ldots, 0) = 0$$

$$k_i^2 - \omega_i^2 + 1 + 2\varepsilon Q_{(E)}(k_i, \omega_i, E_i, 0, 0) = 0$$

$$[\omega] = v[k], \qquad [kE] - v[\omega E] = \varepsilon \int_{E_1}^{E} Q_{(\theta)} \frac{dE}{E'}$$

$$v[E] + (v^2-1)[\omega k E] = -2\varepsilon v \int_{E_1}^{E_2} Q_{(E)} dE - \varepsilon \int_{E_1}^{E_2} \omega Q_{(\theta)} \frac{dE}{E'}$$

If this system admits solutions, then a discontinuity can be realized with the above values. Consequently the dispersive system admits a discrete set of discontinuities. We emphasize that a nonconservative initial system also admits the discontinuity-type solutions, but the discontinuity parameters must then be specified more accurately.

BIBLIOGRAPHY

1. Karpman, V. I., Nonlinear waves in dispersive media. Tr. Novosib. Univ. 1968.
2. Gaponov, A. V., Ostrovskii, L. A. and Rabinovich, M. I., One-dimensional waves in nonlinear systems with dispersion. Izv. vuzov Radiofizika, Vol. 13, №2, 1970.

3. Whitham, G. B., Nonlinear dispersive waves. Proc. Roy. Soc. A, Vol. 283, №1393, 1965.
4. Whitham, G. B., A general approach to linear and nonlinear dispersive waves using a Lagrangian. J. Fluid Mech., Vol. 22, №2, 1965.
5. Venitsianov, E. V., Asymptotic character of the averaged variational principle in the theory of nonlinear waves with dispersion and small dissipation. Dokl. Akad. Nauk SSSR, Vol. 198, №3, 1971.
6. Mitropopol'skii, Iu. A., Problems of the Asymptotic Theory of Nonstationary Oscillations. M., "Nauka", 1964.
7. Sedov, L. I., Mathematical methods of constructing new models of continuous media. Uspekhi matem. nauk, Vol. 20, №5, 1965.
8. Ostrovskii, L. A. and Pelinovskii, E. N., Averaging method for nonsinusoidal waves. Dokl. Akad. Nauk SSSR, Vol. 195, №4, 1970.
9. Whitham, G. B., Variational methods and applications to water waves. Proc. Roy. Soc. A, Vol. 299, №1456, 1967.
10. Lur'e, M. V., Application of a variational principle to investigate discontinuities in a continuum. PMM Vol. 30, №4, 1966.
11. Noether, E., Invariante Variationsprobleme. Nachr. Königl. Gessell. Wiss. Göttingen, Math.-phys. Klasse, S. 235-257, 1918.
12. Gel'fand, I. M., Some problems in the theory of quasilinear equations. Uspekhi matem. nauk, Vol. 14, №2, 1959.

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ON THE STABILITY OF THE EQUILIBRIUM POSITIONS OF A SOLID BODY WITH A CAVITY CONTAINING A LIQUID

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We consider a solid body with a simply-connected cavity containing a liquid. In the case when the potential energy is positive definite with respect to a part of the generalized coordinates, we give sufficient conditions for the asymptotic stability of the equilibrium position relative to a part of the coordinates, to the generalized velocities, and to the kinetic energy of the fluid. It is shown that the asymptotic stability is uniform with respect to initial excitations from any compact set in some neighborhood of the equilibrium position.

1. We consider a system of differential equations of perturbed motion

$$\mathbf{x}^{\cdot} = \mathbf{X}(t, \mathbf{x}) \qquad (\mathbf{X}(t, 0) \equiv 0) \tag{1.1}$$

where $\mathbf{x} = (y_1, \ldots, y_m, \quad z_1, \ldots, z_p)$ is a real n-vector and, $n = m + p$, $m > 0$, $p \geqslant 0$. We assume that